ON THE THEORY OF NONLINEAR VIBRATIONS OF A LIQUID OF FINITE VOLUME

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This paper is devoted to the study of nonlinear vibrations of a finite volume of liquid. However, this work has a number of shortcomings, and therefore, the author deemed it unsuitable for publication for a long time.

Thus, it should be noted that the results are obtained in a formal manner, the convergence of the procedure is not proved, and the computations involved in the application of the procedure are very difficult.

It was not clear whether these results could be extended to the case of vibrations of a body filled with a liquid having a free surface, since a completeness theorem of the principal oscillations of such a body was not available.

Inasmuch as in nonlinear vibrations the amplitude approaches its limiting value very rapidly and the waves disintegrate, it seemed that in practice one needed either a linear theory or a theory taking into account the energy dissipation in wave disintegration.

Until the present time it was not possible to substantiate the procedure. Nevertheless, the following circumstances, shall we say, excuse the present publication of the theory:

1. The application of high speed computers permits of carrying out the required calculations immediately with no special effort, especially since in recent years effective numerical methods for solving the pertinent boundary value problems have been developed.

2. The question of completeness of the principal vibrations of a body with a fluid is completely settled, and the extension of the developed theory to the case of vibrations of a body with a fluid having a free

surface does not present any substantial difficulties.

3. It is in principle impossible to conduct an analysis of resonance phenomena in a liquid within the linear theory. At the present time, however, this problem has become a subject of practical interest.

1. Free Vibrations of the Liquid. 1°. The problem reduces to the determination of the function ϕ which is harmonic in the region τ (see Fig. 1, where the symbols are introduced), bounded by a rigid surface and a free boundary $z = \zeta(x,y,t)$ and satisfying the conditions

$$\frac{\partial \varphi}{\partial n} = 0 \qquad \text{on } \Sigma \tag{1.1}$$

$$\frac{\partial \varphi}{\partial t} + g\zeta + \frac{1}{2}(\nabla\varphi)^2 = 0 \qquad \text{for } z = \zeta \tag{1.2}$$

where the function ζ is determined from the kinematic relation

$$\frac{d\zeta}{dt} = \left(\frac{d\varphi}{dz}\right)_{z=\zeta} \tag{1.3}$$

Denote by λ_n and $\psi_n(x, y)$ the characteristic numbers and values of the integral equation

$$\psi(x, y) = \lambda \int_S H(x, y, 0;\ x', y', 0')\, \psi(x', y')\, ds \tag{1.4}$$

where H is the Green's function of the Neumann problem for the region τ', bounded by the surface Σ and the plane $z = 0$.

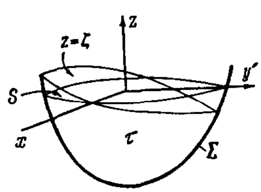

Fig. 1.

The following result is found to hold [4]. Let $\phi_n{}^*$ and $\zeta_n{}^*$ be the velocity potential and the free boundary of the n-th free vibration mode of infinitesimally small amplitude and let

$$\zeta_n{}^* = \psi_n(x, y) \sin \sigma_n t$$

Then

$$\varphi_n{}^* = \sigma_n \Phi_n(x, y, z) \cos \sigma_n t \tag{1.5}$$

Here

$$\Phi_n(x, y, z) = \int_S H(x, y, z; x', y', 0)\, \psi_n(x', y')\, ds$$

i.e.

$$\Phi_n(x, y, 0) = \frac{\psi_n(x, y)}{\lambda_n}$$

The frequencies of natural vibrations σ_n and the numbers λ_n are related by

$$\sigma_n^2 = g\lambda_n \tag{1.6}$$

2° Let ϵ be some parameter, and let

$$\varphi = \varepsilon \sum_{n=0}^{\infty} \varphi_n \varepsilon^n, \qquad \zeta = \varepsilon \sum_{n=0}^{\infty} \zeta_n \varepsilon^n \tag{1.7}$$

Introduce a new independent variable

$$\tau = \frac{t\sigma_m}{1 + \Sigma h_n \varepsilon^n} \tag{1.8}$$

where h_n are constants to be determined.

Substituting the series (1.7) into conditions (1.2) and (1.3) and transforming to the new variable the following system of equations for the determination of the unknown functions ϕ_n and ζ_n is obtained:

$$\left(\frac{\partial \varphi_0}{\partial \tau}\right)_0 + \frac{1}{\sigma_m} g\zeta_0 = 0 \tag{1.9}$$

$$\left(\frac{\partial \varphi_1}{\partial \tau}\right)_0 + \frac{1}{\sigma_m} g\zeta_1 = A_1(\varphi_0, \zeta_0) - \frac{h_1}{\sigma_m} g\zeta_0$$

$$\left(\frac{\partial \varphi_2}{\partial \tau}\right)_0 + \frac{1}{\sigma_m} g\zeta_2 = A_2(\varphi_0, \zeta_0, \varphi_1, \zeta_1, h_1) - \frac{h_2}{\sigma_m} g\zeta_0$$

$$\frac{\partial \zeta_2}{\partial \tau} = \frac{1}{\sigma_m} \left(\frac{\partial \varphi_0}{\partial z}\right)_0 \tag{1.10}$$

$$\frac{\partial \zeta_1}{\partial \tau} = \frac{1}{\sigma_m} \left(\frac{\partial \varphi_1}{\partial z}\right)_0 + B_1(\varphi_0, \zeta_0, h_1)$$

The functions A_i and B_i can be easily computed. The symbol $(\ )_0$ indicates that the function is to be evaluated at $z = 0$.

3° Zero approximation. Taking the partial derivative with respect to τ of the first of the equations of system (1.9) and using the first of the conditions (1.10), we get

$$\left(\frac{\partial^2 \varphi_0}{\partial \tau^2} + \frac{1}{\sigma_m^2} g \frac{\partial \varphi_0}{\partial z}\right)_0 = 0 \tag{1.11}$$

Let

$$\varphi_0 = \sum f_{n0}(t)\,\phi_n(x, y, z), \quad \phi_n(x, y, 0) = \psi_n(x, y)$$

Then

$$\phi_n(x, y, 0) = \int_S H(x, y, 0, x', y', 0) \left(\frac{\partial \psi_n}{\partial z'}\right)_0 ds$$

i.e.

$$\left(\frac{\partial \phi_n}{\partial z}\right)_0 = \lambda_n \psi_n = \frac{\sigma_n^2}{g} \psi_n \tag{1.12}$$

Thus the following equation is obtained for the function f_{n0}:

$$f_{n0}'' + \frac{\sigma_n^2}{\sigma_m^2} f_{n0} = 0$$

By virtue of the arbitrariness of the initial time reading the unique solution of this system having a period of 2π is*:

$$f_{n0} = 0 \quad \text{for } n \neq m, \qquad f_{m0} = c \cos \tau$$

Let $c = ag/\sigma_m$; then

$$\varphi_0 = \frac{ag}{\sigma_m} \cos \tau \varphi_m \tag{1.13}$$

Using the first of equations (1.10) and equation (1.12) the shape of the wave surface is found:

$$\zeta_0 = a \psi_m \sin \tau \tag{1.14}$$

Here a is an arbitrary constant.

4° First Approximation. Differentiating the second of the equations of system (1.9), replacing $\partial \zeta_1 / \partial \tau$ by its value from the second equation of system (1.10), and noting the values of ζ_0 and ϕ_0 at $z = 0$

$$\frac{\partial^2 \varphi_1}{\partial \tau^2} + \frac{1}{\sigma_m^2} g \frac{\partial \varphi_1}{\partial z} = -\frac{h_1}{\sigma_m} ga \cos \tau \psi_m + a^2 \sin 2\tau F_1^{(2)}(x, y)$$

Here $F_1^{(2)}(x, y)$ is an already known function. The solution is to be found in the form

$$\varphi_1 = \sum f_{n1} \phi_n$$

If one assumes that

$$F_1^{(2)}(x, y) = \sum b_{1k}^{(2)} \psi_n$$

then the following system arises for the determination of f_{n1}:

$$\begin{aligned} f_{n1}'' + \frac{\sigma_n^2}{\sigma_m^2} f_{n1} &= a^2 b_{1n}^{(2)} \sin 2\tau \qquad (n \neq m) \\ f_{m1}'' + f_{m1} &= -\frac{h_1}{\sigma_m} ga \cos \tau + a^2 b_{1m}^{(2)} \sin 2\tau \end{aligned} \tag{1.15}$$

* It is assumed that σ_n^2/σ_m^2 is not an integer when $n \neq m$.

It is necessary and sufficient for the existence of a periodic solution of the system (1.15) that $h_1 = 0$. The amplitude a may be arbitrary; without loss of generality it can be put equal to unity.

When f_{n1} are determined, then ϕ_1 and ζ_1 can be computed immediately.

5°. In order to determine the first correction to the frequency h_2, the second approximation should be studied. Repeating the same reasoning as before one arrives at the following condition for the function ϕ_2 at $z = 0$:

$$\frac{\partial^2\varphi_2}{\partial\tau^2} + \frac{1}{\sigma_m^2} g \frac{\partial\varphi_2}{\partial z} = -\frac{h_2}{\sigma_m} g \cos\tau\psi_m + F_2^{(1)}\cos\tau + F_2^{(3)}\cos 3\tau$$

Let

$$\varphi_2 = \sum f_{n2}\phi_n, \qquad F_2^{(1)} = \sum b_{k2}^{(1)}\psi_n, \qquad F_2^{(3)} = \sum b_{k2}^{(3)}\psi_n$$

The following equations are obtained for the functions $f_{n2}(t)$

$$f_{n2}'' + \frac{\sigma_n^2}{\sigma_m^2} f_{n2} = b_{n2}^{(1)}\cos\tau + b_{n2}^{(3)}\cos 3\tau \qquad (n \neq m)$$

$$f_{m2}'' + f_{m2} = \left(-\frac{h_2}{\sigma_m} g + b_{m2}^{(1)}\right)\cos\tau + b_{m2}^{(3)}\cos 3\tau \qquad (n = m)$$

It is necessary and sufficient for the existence of periodic solutions of this system that

$$h_2 = \frac{b_{m2}^{(1)}\sigma_m}{g} \tag{1.16}$$

If equation (1.16) holds, then the required solution is easily found.

6°. It can be easily shown by induction that it is possible to calculate any arbitrary approximation. The condition for the function ϕ_k will be:

for k odd

$$\frac{\partial^2\varphi_k}{\partial\tau^2} + \frac{1}{\sigma_m} g \frac{\partial\varphi_k}{\partial z} = -\frac{h_k}{\sigma_m} g\cos\tau\psi_m + \sum_{s=2}^{k+1} F_k^{(s)}(x, y)\sin s\tau \qquad (s = 2, 4, 6, \ldots)$$

for k even

$$\frac{\partial^2\varphi_k}{\partial\tau^2} + \frac{1}{\sigma_m} g \frac{\partial\varphi_k}{\partial z} = -\frac{h_k}{\sigma_m} g\cos\tau\psi_m + \sum_{s=1}^{k+1} F_k^{(s)}(x, y)\cos s\tau \qquad (s = 3, 5, 7)$$

Let

$$\varphi_k = \sum f_{nk}\phi_n, \qquad F_k^{(3)} = \sum b_{nk}^{(s)}\psi_n$$

The following system of equations is obtained for the functions f_{nk}:

$$f_{nk}^{\bullet\bullet} + \frac{\sigma_n^2}{\sigma_m^2} f_{nk} = \sum b_{nk}^{(s)} \frac{\sin}{\cos} s\tau \qquad (n \neq m)$$

$$f_{mk}^{\bullet\bullet} + f_{mk} = -\frac{h_k}{\sigma_m} g \cos\tau + \sum b_{mk^s} \frac{\sin}{\cos} s\tau \qquad (n=m)$$

Here the sine corresponds to the odd and the cosine to the even values of k (at the same time the index s also takes on even or odd values, respectively).

If follows that if k is odd, then necessarily $h = 0$, if k is even, then

$$h_k = \frac{b_{km}^{(1)}}{g} \tag{1.17}$$

7° Using the obtained solution one can compute the velocity potential and all characteristic flows. In particular, the equation of the free surface can be presented in the following form

$$\zeta = \varepsilon\psi_m(x, y)\sin\frac{\sigma_m t}{1 + h_2\varepsilon^2 + \ldots} + \varepsilon^2(\ldots) + \ldots \tag{1.18}$$

Thus, the parameter ϵ is the amplitude of the wave. A number of other general derivations can be performed:

a) The frequency is a function of the amplitude:

$$\sigma_m^* = \frac{\sigma_m}{1 + Q(\varepsilon, \sigma_m)} \tag{1.19}$$

Thus, the spectrum appears to be not discrete but stepwise continuous.

b) Periodic vibrations with an arbitrary amplitude lying inside the circle of convergence of series (1.18) are possible. This appears as one of the analogies between the vibrations studied and vibrations of conservative systems having a finite number of degrees of freedom.

For a more detailed analysis of the properties of free vibrations, it is necessary to specify the form of the container.

In the particular case of cylindrical containers it can be shown that:

a) It is impossible to find a time t at which the free surface is a plane surface.

b) There exist no stationary nodes.

The proof of the convergence of the proposed procedure involves a number of difficulties. In particular, the realization of the procedure is possible only when σ_n/σ_m is not an integer, provided $n \neq m$.

As a special case the solution of the problem of Sekerzh-Zen'kovich

[5], which investigates standing waves on an infinite fluid, is obtained from these results. His theory is equivalent to the theory of fluid vibrations in a tank having the form of a parallelepiped.

2. Forced vibrations and resonance phenomena. 1°. The problem of forced vibrations of a liquid under the action of a field of mass forces differs from the one discussed above by the fact that condition (1.2) is not homogeneous:

$$\frac{\partial\varphi}{\partial t} + g\zeta + \tfrac{1}{2}(\nabla\varphi)^2 = U(t, x, y) \qquad \text{at } z = 0 \tag{2.1}$$

Assume, for the sake of simplicity, that

$$U = \frac{\mu}{p}\sin pt f(x, y) \tag{2.2}$$

The problem is to find periodic solutions of this system having a period of $2\pi/p$.

2°. For $\mu = 0$ this problem will describe the vibrations of a certain conservative system; therefore, the usual quasi-linear treatment may turn out to be insufficient. The present system will be analysed as a system close to Liapunov's system, and a solution will be sought which goes over to the periodic solution of the problem (1.1)-(1.3) as $\mu \to 0$.

It was established above that periodic solutions of this problem, whose period depends on the amplitude, may formally exist. In that case the period T is given, and it is equal to the period of the external force $2\pi/p$, and consequently, the amplitude should be determined from the relation

$$T = \frac{2\pi}{pn} \tag{2.3}$$

where n is an arbitrary integer.

Using formula (1.19), equation (2.3) can be written as follows:

$$\varepsilon^2 h_2 + \varepsilon^4 h_4 + \ldots = \frac{\sigma_m - pn}{pn} \tag{2.4}$$

It can be seen that the proposed problem is known to have no unique solution; there can exist solutions of period $2\pi/p$, which may go over to trivial solutions as $\mu \to 0$; there may also exist solutions which go over to the nontrivial solutions of the problem analysed in the previous paragraph as $\mu \to 0$. This paragraph will deal only with the finding of solutions of the first type.

3°. The vibrations far away from resonance will be studied. Let

$$\varphi = \sum_1^\infty \varphi_n \mu^n, \qquad \zeta = \sum_1^\infty \zeta_n \mu^n \tag{2.5}$$

Repeating the reasoning of the previous paragraph leads to the following equations on ϕ_n and ζ_n, respectively, at $z = 0$:

$$\frac{\partial \varphi_1}{\partial t} + g\zeta_1 = \frac{1}{p}\sin pt f(x, y), \qquad \frac{\partial \varphi_2}{\partial t} + g\zeta_2 = A_1(\varphi_2, \zeta_1), \ldots \tag{2.6}$$

$$\frac{\partial \zeta_1}{\partial t} = \frac{\partial \varphi_1}{\partial z}, \qquad \frac{\partial \zeta_2}{\partial t} = \frac{\partial \varphi_2}{\partial z} + B_1(\varphi_1, \zeta_1), \ldots \tag{2.7}$$

Differentiating the first of equations (2.5) and using the first of equations (2.7) we get

$$\frac{\partial^2 \varphi_1}{\partial t^2} + g\frac{\partial \varphi_1}{\partial y} = f(x, y)\cos pt \qquad (z = 0)$$

Further, let

$$\varphi_1 = \sum f_{n1}(t)\,\phi_n(x, y, z), \qquad f(x, y) = \sum c_n \psi_n(x, y)$$

The functions f_{n1} will satisfy the following system of equations

$$f_{n1}'' + \sigma_n^2 f_{n1} = c_n \cos pt$$

Therefore,

$$\varphi_1 = \sum \frac{c_n \phi_n}{\sigma_n^2 - p_n^2}\cos pt \tag{2.8}$$

The computation of the last approximations presents no difficulties.

4. The solution in the vicinity of resonance will be studied. The solution in the form (2.8) loses its meaning when $p_n \to \sigma_m$. In order to study the character of the vibrations, the "detuning" $p_n^2 - \sigma_m^2$ will be considered small:

$$p^2 = \sigma_m^2 + \mu\alpha$$

Then using formula (1.5) let

$$g = \frac{p^2}{\lambda_m} + \mu\frac{\alpha}{\lambda_m}$$

It can be shown that for $a = 0$ there exist no periodic solutions of the problem (2.1) of the form

$$\varphi = \sum_1^\infty \varphi_n \mu^{an/b}, \qquad \zeta = \sum_1^\infty \zeta_n \mu^{an/b}$$

provided a and b are integers and $b \neq 3a$. Hence let

$$\varphi = \sum_1^\infty \varphi_n \mu^{1/3\, n}, \qquad \zeta = \sum_1^\infty \zeta_n \mu^{1/3\, n} \tag{2.9}$$

The system of equations (2.6) and (2.7) can in this case be the following $(z = 0)$:

$$\frac{\partial\varphi_1}{\partial t}+\frac{p^2}{\lambda_m}\zeta_1=0, \qquad \frac{\partial\varphi_2}{\partial t}+\frac{p^2}{\lambda_m}\zeta_2=A_1$$

$$\frac{\partial\varphi_3}{\partial t}+\frac{p^2}{\lambda_m}\zeta_3=A_3+\frac{f(x,y)}{p}\sin pt, \ldots$$

$$\frac{\partial\zeta_1}{\partial t}=\frac{\partial\varphi_1}{\partial z}, \qquad \frac{\partial\zeta_2}{\partial t}=\frac{\partial\varphi_2}{\partial z}+B_1, \ldots$$

The functions A_i and B_i appearing in these equations are determined by the formulas $(z = 0)$

$$A_1=-\frac{\partial}{\partial t}\left(\zeta_1\frac{\partial\varphi_1}{\partial z}\right)-\tfrac{1}{2}(\nabla\varphi_1)^2 \tag{2.11}$$

$$A_2=-\frac{\partial}{\partial t}\left(\zeta_1\frac{\partial\varphi_2}{\partial z}+\zeta_1^2\frac{\partial^2\varphi_1}{\partial z^2}+r_2\frac{\partial\varphi_1}{\partial z}\right)-\nabla\varphi_1\left(\zeta_1\frac{\partial}{\partial z}\nabla\varphi_1+\nabla\varphi_2\right), \ldots$$

$$B_1=-\frac{\partial\zeta_1}{\partial x}\frac{\partial\varphi_1}{\partial x}+\zeta_1\frac{\partial^2\varphi_1}{\partial z^2} \tag{2.12}$$

$$B_2=-\frac{\partial\zeta_1}{\partial x}\left(\frac{\partial\varphi_2}{\partial x}+\zeta_1\frac{\partial\varphi_1}{\partial x}\right)-\frac{\partial\zeta_2}{\partial x}\frac{\partial\varphi_1}{\partial x}+\zeta_1\frac{\partial^2\varphi_2}{\partial z^2}+\zeta_1^2\frac{\partial^3\varphi_1}{\partial z^3}+\zeta_2\frac{\partial^2\varphi_1}{\partial z^2}, \ldots$$

Differentiating with respect to t the first of equations (2.9) and using the first of equations (2.10) yields $(z = 0)$

$$\frac{\partial^2\varphi_1}{\partial t^2}+\frac{p^2}{\lambda_m}\frac{\partial\varphi_1}{\partial z}=0 \tag{2.13}$$

Let as before

$$\varphi_1=\sum f_{n1}\phi_n$$

The following system of equations is obtained for the functions f_{n1}:

$$f_{n1}^{\cdot\cdot}+\frac{\lambda_n}{\lambda_m}p^2 f_{n1}=0 \qquad (n=1,2,\ldots) \tag{2.14}$$

The unique periodic solution of the system (2.14) having a period of $2\pi/p$ will be

$$f_{n1}\equiv 0 \quad \text{for } n\neq m, \qquad f_{m1}=M\sin pt+N\cos pt \tag{2.15}$$

Here M and N are constants yet to be determined.

Using formulas (1.6) and the first of the formulas (2.10) we get

$$\varphi_1=\phi_m(x,y,z)(M\sin pt+N\cos pt) \tag{2.16}$$

$$\zeta_1=-\frac{\lambda_m\psi_m(x,y)}{p}(M\cos pt-N\sin pt)$$

Differentiate the second formula of (2.9) partially with respect to t:

$$\frac{\partial^2\varphi_2}{\partial t^2}+\frac{p^2}{\lambda_m}\frac{\partial\varphi_2}{\partial z}=\frac{\partial A_1}{\partial t}-\frac{\partial B_1}{\partial t} \tag{2.17}$$

Here A_1 and B_1 are computed from (2.11) and (2.12), respectively.

After performing the calculations the result is

$$\frac{\partial A_1}{\partial t}-\frac{\partial B_1}{\partial t}=\pi_{11}(M, N)A_{11}(x, y)\sin 2pt+\pi_{12}(M, N)A_{12}(x, y)\cos 2pt \tag{2.18}$$

Here π_{11} and π_{12} are essentially quadratic forms of their variables.

In order to determine the potential let

$$\varphi_2=\sum f_{n2}\phi_n, \qquad A_{ij}=\sum \alpha_{ij}^{(n)}\psi_n$$

Then the following system of equations is obtained for the functions f_{n2}

$$\ddot{f}_{n2}+\frac{\lambda_n}{\lambda_m}p^2 f_{n2}=\pi_{11}\alpha_{11}^{(n)}\sin 2pt+\pi_{12}\alpha_{12}^{(n)}\cos 2pt \quad (n=1, 2, \ldots) \tag{2.19}$$

Inasmuch as the assumption $\lambda_n/\lambda_m \neq k^2$, where k is an integer (if $n \neq m$), holds, the unique periodic solution of the system (2.19) with period $(2\pi/p)$ is

$$f_{n2}=\pi_{11}d_{11}^{(n)}\sin 2pt+\pi_{12}d_{12}^{(n)}\cos 2pt \qquad n\neq m$$

$$f_{m2}=\pi_{11}d_{11}^{(m)}\sin 2pt+\pi_{12}d_{12}^{(m)}\cos 2pt+M_1\sin pt+N_1\cos pt \tag{2.20}$$

Here

$$d_{ik}^{(n)}=\frac{\alpha_{ik}^{(n)}}{p^2[\lambda_n^2/\lambda_m^2]}$$

and the constants M_1 and N_1 are to be determined.

Repeating the sequence of computations demonstrated above, the following equation for the determination of the third approximation ($z = 0$) is obtained:

$$\frac{\partial^2\varphi_3}{\partial t^2}+\frac{p^2}{\lambda_m}\frac{\partial\varphi_3}{\partial z}=\sin pt\sum_{i=0}^{3}a_{i,3-i}(x, y)M^iN^{3-i}+$$

$$+\cos pt\left\{\sum_{i=0}^{3}b_{i,3-i}(x, y)M^iN^{3-i}+f(x, y)\right\}+$$

$$+\sin 2pt\,F_1(x, y, M, N, M_1, N_1)+\ldots+\cos 3pt\,F_4(x, y, M, N, M_1, N_1)$$

The following expressions will be introduced

$$\varphi_3 = \sum_k f_{k3}\phi_k, \qquad a_{ij} = \sum_k \alpha_{ij}^{(k)}\psi_k, \qquad b_{ij} = \sum_k \beta_{ij}^{(k)}\psi_k$$

$$F_s = \sum_k \gamma_{sk}(M, N, M_1, N_1)\psi_k(x, y), \qquad f(x, y) = \sum c_k\psi_k$$

Similar to the system (2.19) a system is obtained whose n-th equation is

$$f_{n3}'' + \frac{\lambda_n}{\lambda_m} p^2 f_{n3} = \sin pt \sum_{i=0}^{3} \alpha_{i,\,3-i} M^i N^{3-i} + \cos pt \left\{\sum_{i=0}^{3} \beta_{i,\,3-i} M^i N^{3-i} + e_n\right\} +$$
$$+ \sin 2pt\gamma_{1n} + \cos 2pt\gamma_{2n} + \sin 3pt\gamma_{3n} + \cos 3pt\gamma_{4n} \tag{2.21}$$

For equation (2.21) to have a periodic solution of period $2\pi/p$ it is necessary and sufficient that the numbers M and N satisfy the following system of equations:

$$\sum_{i=0}^{3} \alpha_{i,\,3-i} M^i N^{3-i} = 0, \qquad \sum_{i=0}^{3} \beta_{i,\,3-i} M^i N^{3-i} + e_m = 0 \tag{2.22}$$

Thus, a system of two cubic equations determining the amplitude and phase of the function ϕ_1 has been arrived at. Function ϕ_2 will contain two constants M_1 and N_1, which should be determined from the fourth approximation.

3. On the Iu.A. Kravtchenko problem. Papers [1-3] study experimentally and theoretically seiche type oscillations of water in ports induced by waves coming from the open sea.

It is assumed that the port basin has the shape of a cylinder or a parallelepiped and is connected to the open sea by a channel. It is assumed that the waves, while propagating through the channel, remain intact with their parameters unchanging. It is necessary to determine the character of the waves appearing on the surface inside the port. This problem differs from problem (1.1)-(1.3) by the fact that the condition (1.1) will be replaced by the following one (Fig. 2):

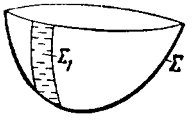

Fig. 2.

$$\frac{\partial\varphi}{\partial n} = F(P)\cos pt \quad \text{for } P \in \Sigma_1$$
$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{for } P \in \Sigma - \Sigma_1 \tag{3.1}$$

This problem can be reduced to the problem studied in Section 2.

To accomplish this it is sufficient to let $\phi = \phi_1 + \phi_2$, where ϕ_1 is an arbitrary function harmonic in r, satisfying conditions (3.1). Then on Σ the function ϕ_2 will satisfy the condition $\partial \psi_2/\partial n = 0$.

The condition that the pressure be constant is rewritten as follows:

$$\frac{\partial \varphi_2}{\partial t} + g\zeta + \frac{1}{2}(\nabla\varphi)^2 = \mu F(t)\,\Phi(\mu, \varphi_1, \varphi_2) \tag{3.2}$$

where the function Φ is expressed in terms of ϕ_1 and ϕ_2.

The kinematic condition is

$$\frac{d\zeta}{dt} = \left(\frac{\partial \varphi_1}{\partial z} + \frac{\partial \varphi_2}{\partial z}\right)_{z=\zeta} \tag{3.3}$$

The procedure outlined in Section 2 can be applied to generate the function and to determine the resonance region.

In conclusion a few remarks will be made regarding the difficulties encountered in the construction of a nonlinear theory and the desirable directions further research should take.

The author performed a series of experiments for the verification of the theory of forced oscillations of pendulums having a cavity in the form of a cylinder or a parallelepiped. These experiments verified quantitatively the analogy with the nonlinear vibrations of mechanical systems having a finite number of degrees of freedom, which is discussed in this paper.

However, when doing this, new cases were discovered. Let, for instance, the frequency of the external force be p and let it steadily increase, approaching from the left the value σ_1 (Fig. 3). Then, if the difference $\sigma_1 - p$ is not very small, the dependence of the amplitude A on the frequency of forced vibrations follows the curve Γ_1 fairly well. If this difference is small then the change of amplitude follows the curve Γ_2.

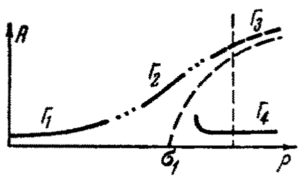

Fig. 3.

If the complete analogy with the system with a finite number of degrees of freedom were true then the variation of $A(p)$ should follow the dotted curve Γ_3. Furthermore, this region would be unstable and the system would go over to the region described by curve Γ_4. According to the above discussion this would already be a second mode vibration. This fact is not found in the experiments. The system never went to the region Γ_3. Already for $p > \sigma_1$ and sufficiently close to σ_1 the Stokes limiting amplitude was reached and the waves disintegrated (for high frequency vibrations this disintegration occurred even sooner, when $p \leqslant \sigma_1$). This disintegration leads to an irreversible loss of energy. Therefore, in order to investigate the vibrations of real systems in regions close to resonance (for instance, in the theory of flutter of a wing carrying fuel tanks) it is necessary to create a model which would take into account the possibility of wave disintegration and the increase of entropy.

Because of the complexity of the computations of the problems studied it is necessary to use approximate schemes which, however, still take into account the essentially nonlinear nature of the phenomenon. It should be noted that in many cases it is necessary to analyse only the first natural modes where the shape of the surface has a small curvature. Because of this it is obviously expedient here to develop methods analogous to the variational methods of Lavrent'ev.

In the nonlinear form this problem is almost unexplored. The basic question of the existence of periodic solutions of this problem still remains open.

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